Part 2: Question 1 (20 points)

Consider a pure exchange economy with L goods, I consumers and one firm whose production technology is free disposal $Y_1 = -\mathbb{R}^L_+$. Each consumer i's preferences $\succeq_i$ satisfy completeness, transitivity, continuity, strict convexity, and strong monotonicity. The consumers' consumption sets are $X_i = \mathbb{R}^L_+$. The aggregate endowment vector satisfies $\bar{\boldsymbol{\omega}} \gg \mathbf{0}$ and each consumer owns an equal share in the firm.

In class we proved that for an economy satisfying these properties, if we allocated the initial endowment such that $\boldsymbol{\omega}_i \gg \mathbf{0}$ for all i, it is guaranteed to have an equilibrium. You may use this fact below without proving it.

(i) [**7 points:**] Show that an allocation $(\boldsymbol{x}, \boldsymbol{y})$ satisfying $\boldsymbol{x}_i \succeq_i \boldsymbol{x}_j$ for all $i, j \in \{1, \ldots, I\}$[1], does not necessarily imply the allocation is Pareto optimal.

Solution:

We just need to come up with one example that satisfies the properties of this economy that isn't Pareto optimal. Suppose $I = L = 2$. The center of the Edgeworth box (where $\boldsymbol{x}_1 = \boldsymbol{x}_2$) satisfies this property. However, unless preferences are identical, the two consumers' indifference curves at this point will form a lens. Thus, it is not Pareto optimal.

Another example would be $\boldsymbol{x}_i = \mathbf{0}$ for all consumers. This clearly satisfies the property, but it is not Pareto optimal (because all consumers have monotonic preferences and we are not consuming the full endowment which satisfies $\bar{\boldsymbol{\omega}} \gg \mathbf{0}$).

(ii) [**6 points:**] If the initial endowment was allocated such that $\boldsymbol{\omega}_i \gg \mathbf{0}$ for all i, we know from above that an equilibrium is guaranteed to exist. Denote the allocation and price vector in equilibrium as $(\boldsymbol{x}^\star, \boldsymbol{y}^\star, \boldsymbol{p})$. Prove that this equilibrium allocation is Pareto optimal.

Solution:

This is exactly the proof of the first fundamental welfare theorem. For completeness I reproduce the proof below.

1. Because $(\boldsymbol{x}^\star, \boldsymbol{y}^\star, \boldsymbol{p})$ is an equilibrium, if $\boldsymbol{x}_i \succ_i \boldsymbol{x}_i^\star$, then $\boldsymbol{p} \cdot \boldsymbol{x}_i > \boldsymbol{p} \cdot \boldsymbol{\omega}_i \equiv w_i$.
2. Furthermore, if $\boldsymbol{x}_i \succeq_i \boldsymbol{x}_i^\star$, then $\boldsymbol{p} \cdot \boldsymbol{x}_i \geq w_i$.
   - Suppose there is an $\boldsymbol{x}_i' \neq \boldsymbol{x}_i^\star$ satisfying $\boldsymbol{x}_i' \succeq_i \boldsymbol{x}_i^\star$ but $\boldsymbol{p} \cdot \boldsymbol{x}_i' < w_i$.
   - By strict convexity there is an allocation $\boldsymbol{x}_i'' = \frac{1}{2}\boldsymbol{x}_i^\star + \frac{1}{2}\boldsymbol{x}_i'$ satisfying $\boldsymbol{x}'' \succ_i \boldsymbol{x}_i^\star$ and $\boldsymbol{p} \cdot \boldsymbol{x}_i'' < w_i$, which contradicts 1.

[1] In words: in the allocation, no consumer strictly prefers another consumer's bundle to their own.

- Therefore it must be that $\boldsymbol{x}_i \succeq_i \boldsymbol{x}_i^\star$, then $\boldsymbol{p} \cdot \boldsymbol{x}_i \geq w_i$.

3. Suppose $\exists\,(\boldsymbol{x}', \boldsymbol{y}')$ that Pareto dominates $(\boldsymbol{x}^\star, \boldsymbol{y}^\star)$.
   - By (1) & (2), $\boldsymbol{p} \cdot \boldsymbol{x}_i' \geq w_i$ $\forall$i and $\boldsymbol{p} \cdot \boldsymbol{x}_i' > w_i$ for at least one i.
   - So $\sum_{i=1}^{I} \boldsymbol{p} \cdot \boldsymbol{x}_i' > \sum_{i=1}^{I} w_i = \boldsymbol{p} \cdot \bar{\boldsymbol{\omega}} + \boldsymbol{p} \cdot \boldsymbol{y}_1^\star$.
4. Because $\boldsymbol{y}_1^\star$ is profit-maximizing at $\boldsymbol{p}$, we have $\boldsymbol{p} \cdot \boldsymbol{y}_1^\star \geq \boldsymbol{p} \cdot \boldsymbol{y}_1$ $\forall \boldsymbol{y}_1 \in Y_1$.
   - Therefore $\boldsymbol{p} \cdot \bar{\boldsymbol{\omega}} + \boldsymbol{p} \cdot \boldsymbol{y}_1^\star \geq \boldsymbol{p} \cdot \bar{\boldsymbol{\omega}} + \boldsymbol{p} \cdot \boldsymbol{y}_1'$.
5. Because $(\boldsymbol{x}', \boldsymbol{y}')$ is Pareto improving: $\sum_{i=1}^{I} \boldsymbol{x}_i' = \bar{\boldsymbol{\omega}} + \boldsymbol{y}_1'$.
   - This implies $\sum_{i=1}^{I} \boldsymbol{p} \cdot \boldsymbol{x}_i' = \boldsymbol{p} \cdot \bar{\boldsymbol{\omega}} + \boldsymbol{p} \cdot \boldsymbol{y}_1'$
6. But (3) & (4) imply $\sum_{i=1}^{I} \boldsymbol{p} \cdot \boldsymbol{x}_i' > \boldsymbol{p} \cdot \bar{\boldsymbol{\omega}} + \boldsymbol{p} \cdot \boldsymbol{y}_1'$.
   - But this contradicts (5).

(iii) [**7 points:**] Prove that an allocation $(\boldsymbol{x}, \boldsymbol{y})$ that satisfies both Pareto optimality and $\boldsymbol{x}_i \succeq_i \boldsymbol{x}_j$ for all $i, j \in \{1, \ldots, I\}$ exists in this economy.

Solution:
The allocation associated with the Walrasian equilibrium arising from $\boldsymbol{\omega}_i = \frac{1}{I}\bar{\boldsymbol{\omega}}$ for all i satisfies both Pareto optimality and $\boldsymbol{x}_i \succeq_i \boldsymbol{x}_j$ for all $i, j \in \{1, \ldots, I\}$.

- As stated in the problem introduction, all the assumptions required for the existence of an equilibrium are satisfied ($\boldsymbol{\omega}_i \gg \mathbf{0}$ for all i and preferences are continuous, strictly convex and monotone).
- Given equilibrium prices, all consumers have the exact same budget set. Thus their optimal choice in equilibrium must be at least as good as the choice of all other consumers. So the allocation resulting from the equilibrium satisfies $\boldsymbol{x}_i \succeq_i \boldsymbol{x}_j$ for all $i, j \in \{1, \ldots, I\}$.
- From (ii) we also showed the equilibrium is Pareto efficient. Thus it satisfies both properties.

Part 2: Question 2 (20 points)

Consider an economy with $L = 3$ goods. There is one consumer with the utility function:

$$u = v_1(x_1) + v_2(x_2) + x_3$$

where $v_\ell'(x_\ell) > 0$ and $v_\ell''(x_\ell) < 0$ for all $x_\ell \geq 0$ and $\ell = 1, 2$. The consumer must consume nonnegative quantities of goods 1 and 2 but can consume any quantity of good 3. The consumption set is therefore $X = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$. The consumer has an endowment $\omega_3 > 0$ of good 3 and none of goods 1 and 2 ($\omega_1 = \omega_2 = 0$). The prices of goods 1 and 2 are p_1 and p_2 and the price of good 3 is normalized to 1.

There are two price-taking firms owned by the single consumer whose production sets are:

$$\mathcal{Y}_1 = \left\{ \boldsymbol{y}_1 \in \mathbb{R}^3 : y_{11} \le f_1\left(-y_{31}\right), y_{21} = 0, y_{31} \le 0 \right\}$$
$$\mathcal{Y}_2 = \left\{ \boldsymbol{y}_2 \in \mathbb{R}^3 : y_{12} = 0, y_{22} \le f_2\left(-y_{32} - e\left(y_{11}\right)\right), y_{32} + e\left(y_{11}\right) \le 0 \right\}$$

where $f_j'(\cdot) < 0$ and $f_j''(\cdot) > 0$ for $j = 1,2$ for all $y_{3j} \le 0$. This means that a higher input $-y_{3j}$ leads to more output, but at a decreasing rate. Notice that firm 1 has a negative production externality on firm 2. It essentially acts as a fixed cost for firm 2. Firm 2 needs to use $e\left(y_{11}\right)$ units of the input good 3 to "pay for" the externality before it can use additional units of the input to product good 2. The externality satisfies $e'\left(y_{11}\right) > 0$ for all $y_{11} \ge 0$.

Define the following functions:

$$c_1\left(y_{11}\right) = \min_{-y_{31} \ge 0} \left\{ -y_{31} \text{ subject to } f_1\left(-y_{31}\right) \ge y_{11} \right\}$$
$$c_2\left(y_{22}\right) = \min_{-y_{32} \ge e(y_{11})} \left\{ -y_{32} \text{ subject to } f_2\left(-y_{32} - e\left(y_{11}\right)\right) \ge y_{22} \right\}$$

where $c_j'(\cdot) > 0$ and $c_j''(\cdot) > 0$. Putting these together we can write the firms' profit functions as

$$\pi_1\left(p_1\right) = \max_{y_{11} \ge 0} \left\{ p_1 y_{11} - c_1\left(y_{11}\right) \right\}$$
$$\pi_2\left(p_2\right) = \max_{y_{22} \ge 0} \left\{ p_2 y_{22} - c_2\left(y_{22}\right) - e\left(y_{11}\right) \right\}$$

Because the consumer owns both firms, the consumer's wealth is given by $\omega_3 + \pi_1\left(p_1\right) + \pi_2\left(p_2\right)$.

Assume that $v_\ell'(0) > c_\ell'(0)$ for $\ell = 1,2$ and that $v_1'(0) > c_1'(0) + e_1'(0)$ such that all solutions are interior. You do not need to consider boundary cases in your solution.

(i) **[5 points]** Characterize the market equilibrium in this economy, assuming that each firm maximizes its own profits (i.e. the single consumer that owns the firms does not dictate their production, but only collects the profits at the end).

Solution:

The consumer sets x_ℓ such that $v_\ell'\left(x_\ell\right) = p_\ell$, for $\ell = 1,2$ and consumes $x_3 = \omega_3 + \pi_1\left(p_1\right) + \pi_2\left(p_2\right) - p_1 x_1 - p_2 x_2$ of good 3. Firm j sets $p_j = c'\left(y_{jj}\right)$ for $j = 1,2$. In equilibrium then:

- $x_1 = y_{11}$ and solves $v_1'(x_1) = c_1'(x_1)$.
- $x_2 = y_{22}$ and solves $v_2'(x_2) = c_2'(x_2)$.
- $x_3 = \omega_3 + y_{13} + y_{23}$ and solves $x_3 = \omega_3 - c_1(x_1) - c_2(x_2) - e(x_1)$.

(ii) [**5 points**] Characterize the Pareto optimal level of output of each firm assuming the social planner seeks to maximize the utility of the single consumer. Show that this is different to part (i).

Solution:
The social planner has the following problem:

$$\max_{\boldsymbol{x}\in X, \boldsymbol{y}\in \mathcal{Y}} v_1(x_1) + v_2(x_2) + x_3 \quad \text{subject to } \boldsymbol{x} = \boldsymbol{\omega} + \boldsymbol{y}$$

Because utility is increasing in all goods, all constraints bind and we will have $x_1 = y_{11}$, $x_2 = y_{22}$ and:

$$x_3 = \omega_3 - c_1(x_1) - c_2(x_2) - e(x_1)$$

The problem is then:

$$\max_{\boldsymbol{x}\in X} v_1(x_1) + v_2(x_2) + \omega_3 - c_1(x_1) - c_2(x_2) - e(x_1)$$

The social planner solution is:

- $x_1 = y_{11}$ and solves $v_1'(x_1) = c_1'(x_1) + e'(x_1)$.
- $x_2 = y_{22}$ and solves $v_2'(x_2) = c_2'(x_2)$.
- $x_3 = \omega_3 + y_{13} + y_{23}$ and solves $x_3 = \omega_3 - c_1(x_1) - c_2(x_2) - e(x_1)$.

This differs from (i) in that the optimal consumption of good 1 considers the production externality on firm 2.

(iii) [**5 points**] Suppose there is a government that can impose a per-unit tax on the production of good 1. What should the tax be set at in order to achieve the Pareto optimal solution in part (ii)?

Solution:
With a per-unit tax τ on the production of good 1, firm 1's problem becomes:

$$\max_{y_{11}\geq 0} p_1 y_{11} - c_1(y_{11}) - \tau y_{11}$$

The first-order condition becomes $p_1 = c_1'(y_{11}) + \tau$. Let y_1° be the optimal production of good 1 from the social planner's problem in (ii). With a per-unit tax on the production of good 1 equal to $\tau = e'(y_1^\circ)$ we can achieve the Pareto optimal solution. This is because the firm's optimality condition becomes $p_1 = c_1'(x_1) + e'(y_1^\circ)$. Combined with the consumer's optimality condition $p_1 = v'(x_1)$ we arrive at an equilibrium condition of $v_1'(y_1) = c_1'(y_1) + e'(y_1^\circ)$, which yields $y_1 = y_1^\circ$.

(iv) [**5 points**] Suppose now the government did not know the cost functions, nor the cost of the externality. They instead passed the following law. Before production takes place, both firms are required to publicly announce a suggested per-unit tax rate $t_j \geq 0$, $j = 1, 2$, on the output of good 1. Then, after production takes place, the following occurs:

- Firm 1 pays the tax suggested by firm 2 (i.e. $t_2 x_1$) to the government.
- Firm 2 receives compensation from the government based on the tax rate suggested by firm 1 (i.e. $t_1 x_1$).
- Both firms are required to pay a tax equal to $(t_2 - t_1)^2$ to the government.

Show that it is optimal for both firms to suggest the same tax, and this tax equals the one from part (iii).

Suggested steps: Think of this as a sequential game with 2 stages and solve it with backward induction. In stage 2, firms choose the optimal production levels given the tax rates from stage 1. In stage 1, firms choose the optimal tax rates knowing that they will optimize given those tax rates in stage 2 (i.e. firm 2 takes into account that it can impact y_{11} through t_2).

Solution:

We solve the game backwards. Given two tax rates, t_1 and t_2, the firms' problems become:

$$\max_{y_{11} \geq 0} \left\{ p_1 y_{11} - c_1(y_{11}) - t_2 y_{11} - (t_1 - t_2)^2 \right\}$$

$$\max_{y_{22} \geq 0} \left\{ p_2 y_{22} - c_2(y_{22}) - e(y_{11}) + t_1 y_{11} - (t_1 - t_2)^2 \right\}$$

Firm 1 chooses y_{11} according to $p_1 = c_1'(y_{11}) - t_2$ and firm 2 chooses y_{22} according to $p_2 = c_2'(y_{22})$. Given this, we now consider the choice of tax rates. Firm 1's problem is straightforward:

$$\max_{t_1 \geq 0} p_1 y_{11} - c_1(y_{11}) - t_2 y_{11} - (t_1 - t_2)^2$$

Firm 1's dominant strategy is to set $t_1 = t_2$. Firm 2's problem is, where we note the dependence of y_{11} on t_2 with $y_{11}(t_2)$:

$$\max_{t_2 \geq 0} p_2 y_{22} - c_2(y_{22}) - e(y_{11}(t_2)) + t_1 y_{11}(t_2) - (t_1 - t_2)^2$$

Taking first-order conditions:

$$-e'(y_{11}(t_2))\, y_{11}'(t_2) + t_1 y_{11}'(t_2) + 2(t_1 - t_2) = 0$$

In equilibrium, with $t_1 = t_2$ from firm 1's problem, we get:

$$\begin{aligned} -e'(y_{11}(t_2))\, y_{11}'(t_2) + t_1 y_{11}'(t_2) + 2\underbrace{(t_1 - t_2)}_{=0} &= 0 \\ [-e'(y_{11}(t_2)) + t_1]\underbrace{y_{11}'(t_2)}_{>0} &= 0 \\ -e'(y_{11}(t_2)) + t_1 &= 0 \\ t_1 = t_2 &= e'(y_{11}(t_2)) \end{aligned}$$

Therefore both firms choose the same tax, and this corresponds to the one in the planner's problem.